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Microscopic Space Dimensions and the Discreteness of Time

Abstract

We present a simple dynamical systems model for the effect of invisible space dimensions on the visible ones. There are three premises. A: Orbits consist of flows of probabilities [P].which is the case in the setting of quantum mechanics. B: The orbits of probabilities are induced by (continuous time) differential or partial differential equations. C: Observable orbit are flow of marginal probabilities where the invisible space variables are averaged out. A theorem is presented which proves that under certain general conditions the transfer of marginal probabilities cannot be achieved by continuous time dynamical systems acting on the space of observable variables but that it can be achieved by discrete time dynamical systems.

1. Introduction:

The realm of general relativity consists of massive structures and great distances while the realm of quantum mechanics consists of the smallest structures such as photons and quarks. In most circumstances one or the other theory applies without conflict. However, in extreme situations, such as black holes, both theories are needed for accurate theoretical analysis. Unfortunately, on very small scales, these two theories are utterly incompatible.

The assumption of space and time continuity together with the machinery of calculus are the pillars on which the theories of quantum mechanics and general relativity rest. Each of these theories has its natural realm; quantum mechanics (QM), the very small scales, and general relativity (GR), the very large. But there are situations where the two must coexist, such as in black holes. As subatomic particles possess mass and space possesses structure on all scales, its has been a great challenge for many decades to apply GR on very small scales where the smooth spatial structures of large scales give way to precipitous landscapes. The inability to predict dynamics on these tiny scales is captured by the ineluctable Heisenberg Principle. It is this fact of nature that forces upon us the reality that point orbits are meaningless, that the only meaningful dynamic trajectories on these scales are those of probabilities which, as the scales become larger , are supported on narrower segments of spacetime, thereby delineating accurately trajectories of points.

As our objective is to present an idea toward the melding of QM and GR we shall deal exclusively with orbits of probabilities. We will present an idea in the form of an example. The idea is to simply untether time from the constraint of continuity. The motivation for this is based on the fact that the constraints of continuity on time and space are so great that only very restricted, scale dependent behavior is possible. If the modern theory of nonlinear dynamics

(chaos) has revealed anything it is that the range of behavior of even the simplest discrete time systems is incredibly rich and, as we shall argue, rich enough to accommodate the behavior of particles in extreme situations where the effects of gravitational attraction on particles must be taken into account.

In order to capture the larger range of dynamical behavior needed in extreme situations, string theory purportedly resolves the incompatibility problem by modifying the equations of general relativity on small scales. But there is a price for this accomplishment in that - to account accurately for quantum effects - space is attributed to have, not three, but nine dimensions. Three dimensions are visible, while the other six dimensions are curled up in tiny, essentially invisible, strings. The 'extra' dimensions provide the additional freedom needed to model the dynamics of a unified QM and GR theory. It is interesting that, in the course of effecting the long sought unification of QM and GR, string theory itself points in the direction of discrete space and time.

It is the objective of this note to suggest that the breaking of the shackles of continuous time at the very outset can yield new insights. In this note we present a simple dynamical system model for the effect of invisible space dimensions on the visible ones. There are three premises on which our model is based:

- A. Orbits consist of flows of probabilities $[P]$.which is the case in the setting of quantum mechanics.
- B. The orbits of probabilities are induced by (continuous time) differential or partial differential equations.
- C. Observable orbit are flow of marginal probabilities where the invisible space variables are averaged out.

In section 2 we present the framework and notation of this note. In section 3 we state our main result, which is a dynamical equivalence between additional space dimensions and the discreteness of time. In section 4 we present 2 examples.

2. Framework and Notation

It is now common knowledge that even simple one dimensional maps have the ability to describe very complicated dynamical behavior of biological and mechanical systems. Modeling dynamics by a map offers much more variety of behavior than do differential equations whose solutions are greatly restricted by time continuity and cannot exhibit chaotic behavior in low dimensions. Describing dynamical behavior by iterating a map, which can arise as a Poincaré section or by direct modeling offers many benefits from an analysis perspective. Once a map is determined, the long term statistical behavior is described by a probability density function, which can be obtained by measurement of the system or by mathematical means using the Frobenius-Perron operator [?] as follows: let $I = [0, 1]$ denote the state space of a dynamical system and let $\tau : I \rightarrow I$, describe the dynamics of the system. The dynamics is described by a probability density function f associated with the unique (absolutely continuous invariant) measure μ . This is stated mathematically by the following equation:

$$\int_A f dx = \int_{\tau^{-1}A} f dx$$

for any (measurable) set $A \subset \mathbb{R}$. The Frobenius-Perron operator, $P_\tau f$, acts on the space of integrable functions and is defined by

$$\int_A f dx = \int_A P_\tau f dx$$

The operator P_τ transforms a pdf into a pdf under the transformation τ . If τ is piecewise smooth and piecewise differentiable on a partition of n subintervals, then we have the following representation for P_τ [?, Chapter 4]:

$$P_\tau f(x) = \sum_{z \in \{\tau^{-1}(x)\}} \frac{f(z)}{|\tau'(z)|} \quad (1)$$

where for any x , the set $\{\tau^{-1}(x)\}$ consists of at most n points. The fixed points of P_τ are the pdf's.

3. Main Result

Let $\text{Pr}_n : R^n x R^m \rightarrow R^n$ be the orthogonal projection.

Theorem 1 *Let X be a bounded subdomain of $R^n x R^m$ and let $\Phi_t(x)$ be a continuous dynamical system on X (such as generated by an ordinary or partial differential equation). Let F be a compact, connected subset of X and let $f(x)$ be a C^1 probability density function supported on F . We assume that at time $t = 1$, the dynamical system $\Phi_t(x)$ transfers $f(x)$ to the probability density function $g(x)$ which has support G and such that $\text{Pr}_n(G) \setminus \text{Pr}_n(F) \neq \emptyset$. We assume additionally that both $\text{Pr}_n(G)$ and $\text{Pr}_n(F)$ are convex. Let $f^*(y)$ and $g^*(y)$ be the marginals of $f(x)$ and $g(x)$ on R^n . Then no continuous time n -dimensional dynamical system on $\text{Pr}_n(G)$ can transfer f^* to g^* , but this can be achieved by an n -dimensional discrete time nonlinear dynamical system.*

Proof. A continuous time dynamical system on $\text{Pr}_n(G)$ which would transfer f^* to g^* , would induce a homeomorphism $h : \text{Pr}_n(G) \rightarrow \text{Pr}_n(G)$, (onto), such that $h(\text{Pr}_n(F)) = \text{Pr}_n(G)$. Since $\text{Pr}_n(G) \setminus \text{Pr}_n(F) \neq \emptyset$ this is impossible. Now, we will construct a nonlinear transformation $T : \text{Pr}_n(G) \rightarrow \text{Pr}_n(G)$, which transfers f^* to g^* . We can represent $\text{Pr}_n(G)$ as a union of disjoint sets $\text{Pr}_n(F) \cup F_1 \cup \dots \cup F_k$ where each of the summands is C^1 diffeomorphic to $\text{Pr}_n(G)$. Let $h_0 : \text{Pr}_n(F) \rightarrow \text{Pr}_n(G)$ be the diffeomorphism such that $h_*(f^* \lambda) = g^* \lambda$ (Lemma 1). Let $h_i : F_i \rightarrow \text{Pr}_n(G)$, $1 \leq i \leq k$, be a diffeomorphism. Let T be a map glued from the diffeomorphisms h_i , $0 \leq i \leq k$. Obviously $T_*(f^* \lambda) = g^* \lambda$, or $P_T f^* = g^*$. ■

Lemma 2 *Let F and G be convex bounded subsets of R^n . Let f and g be normalized C^1 densities on F and G , respectively. Then there exists a C^1 diffeomorphism $h : F \rightarrow G$ such that $h_*(f \lambda) = g \lambda$, where λ is Lebesgue measure on R^n .*

Proof. This follows by Lemma 2 of [M]. ■

Note that if the invisible variables are essential to the dynamical system then the flow of marginals should not be achievable by any dynamical system operating in the space X alone.

In the next section we shall consider two dimensional dynamical systems, that is, a family of two dimensional diffeomorphisms, $\Phi_t(x, y)$, which we may consider to be induced by a deterministic system such as a differential equation or partial differential equation. By a trajectory of Φ , we mean a flow in time of probabilities or probability density functions. We assume we can only observe the x variable and that the y variable acts in an invisible dimension. In terms of the orbit, this means that we observe only the marginal probabilities in x . We then ask: does there exist a one dimensional dynamical system, that is, a family of one dimensional homeomorphisms, $\varphi_t(x)$, which induces the orbit of marginal probabilities. We present an example to show that this is in general impossible. That is, the invisible variable is essential in describing the observed system and no one dimensional continuous time dynamical system can account for the observed one dimensional orbit. However, if we restrict the orbit to a discrete set of times, the example shows that the discrete time orbit of marginal probabilities can be achieved by a nonlinear dynamical system. This suggests a duality between what can be achieved by invisible space dimensions and nonlinear maps acting in discrete time. The need for discrete time is not surprising as it is an ineluctable consequence of string theory itself.

1 Examples

1) Consider the 2 dimensional differential equation:

$$\begin{aligned} x' &= y - \frac{1}{2} \\ y' &= -x + \frac{1}{2} \end{aligned} \tag{2}$$

on the unit square $S = [0, 1] \times [0, 1]$. This differential equation induces a dynamical system $\Phi_t(x, y)$ which is a rotation around the point $(\frac{1}{2}, \frac{1}{2})$ and is given by

$$\Phi_t(x, y) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x - \frac{1}{2} \\ y - \frac{1}{2} \end{pmatrix} + \left(\frac{1}{2}, \frac{1}{2} \right). \tag{3}$$

Let us now define a probability density function on S as follows: $f_0(x, y) = 2$ if $(x, y) \in [0, 0.5] \times [0, 1]$ and $f_0(x, y) = 0$ if $(x, y) \in (0.5, 1] \times [0, 1]$. Let $t = \pi/2$. Then $\Phi_{\pi/2}(x, y)$ transforms $f_0(x, y)$ to $f_1(x, y) = 2$ if $(x, y) \in [0, 1] \times [0.5, 1]$ and $f_1(x, y) = 0$ if $(x, y) \in [0, 1] \times [0, 0.5]$. Let us now consider the associated marginal probability density functions:

$$g_0(x) = \int_0^1 f_0(x, y) dy$$

$$g_1(x) = \int_0^1 f_1(x, y) dy$$

Then $g_0(x) = 2$ if $x \in [0, .5]$ and $g_0 = 0$ if $x \in (.5, 1]$. Also, $g_1(x) = 1$ if $x \in [0, 1]$. Now there is no homeomorphism that transforms g_0 to g_1 , that is, there is no one-dimensional continuous time dynamical system that can transform g_0 to g_1 . However, the triangle map can do it in one iteration as we now show. Let $T : [0, 1] \rightarrow [0, 1]$ be defined by $T(x) = 1 - 2|x - 1/2|$. Its Frobenius-Perron

Operator is given by

$$P_T f(x) = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(\frac{2-x}{2}\right) \quad (4)$$

It is easy to see that $P_T g_0(x) = g_1(x)$.

2) Long thin rod example.

Let $A > 0$ be a fixed number. We consider the rectangular $S = [0, A] \times [0, 1/A]$ as a thin long rod. Consider the 2 dimensional differential equation:

$$\begin{aligned} x' &= A^2 y - \frac{1}{2} A \\ y' &= -A^{-2} x + \frac{1}{2A} \end{aligned} \quad (5)$$

on the rectangular S . This differential equation induces a dynamical system $\Phi_t(x, y)$ which is a distorted rotation around the center of S and is given by

$$\Phi_t(x, y) = \begin{pmatrix} \cos t & \sin t \\ -\frac{\sin t}{A^2} & \frac{\cos t}{A^2} \end{pmatrix} \begin{pmatrix} x - \frac{1}{2}A \\ y - \frac{1}{2A} \end{pmatrix} + \left(\frac{1}{2}A, \frac{1}{2A}\right). \quad (6)$$

Let us again define a probability density function on S as follows: $f_0(x, y) = 2$

if $(x, y) \in [0, A/2] \times [0, 1/A]$ and $f_0(x, y) = 0$ if $(x, y) \in (A/2, A] \times [0, 1/A]$. Let $t = \pi/2$. Then $\Phi_{\pi/2}(x, y)$ transforms $f_0(x, y)$ to $f_1(x, y) = 2$ if $(x, y) \in [0, A] \times [0.5/A, 1/A]$ and $f_1(x, y) = 0$ if $(x, y) \in [0, A] \times [0, 0.5/A]$. Let us now consider the associated marginal probability density functions:

$$g_0(x) = \int_0^A f_0(x, y) dy$$

$$g_1(x) = \int_0^A f_1(x, y) dy$$

Then $g_0(x) = 2/A$ if $x \in [0, A/2]$ and $g_0(x) = 0$ if $x \in (A/2, A]$. Also, $g_1(x) = 1/A$ if $x \in [0, A]$.

Now there is no homeomorphism of $[0, A]$ onto itself that transforms g_0 to g_1 , that is, there is no one-dimensional continuous time dynamical system that can transform g_0 to g_1 . However, an analog of the triangle map can do it in one iteration as we now show. Let $T : [0, A] \rightarrow [0, A]$ be defined by

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq A/2 \\ 2A - 2x & A/2 < x \leq A \end{cases}.$$

Its Frobenius-Perron Operator is given by

$$P_T f(x) = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(A - \frac{x}{2}\right) \quad (7)$$

It is easy to see that $P_T g_0(x) = g_1(x)$.

Summary:

There is no continuous dynamical system in observable variables that can account for behavior. However, we can find a discrete time chaotic dynamical system in the observable dimensions that completely accounts for it.

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In [P] Prigogine argues against the notion of deterministic point trajectories in dynamical systems, proposing that trajectories of ensembles are better models of reality. His arguments are in part due to the fact that it is impossible to specify points with complete accuracy. This is of course in full accord with quantum mechanics where the object of interest is the probability amplitude as it is derived from the wave function. This is also very much the case for chaotic dynamical systems due to the instability of orbits leading to a statistical description of dynamics [BG].